POLITECNICO
MILANO 1863

Course	Phase Kickback
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Academic Year	2025/26

1 Unitaries and Eigenstates

Unitaries represent the operations through which the state of qubits is manipulated. For a single qubit, a generic unitary U is a 2×2 complex-valued matrix with unitary properties, which are:

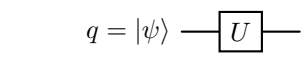
- $U^\dagger = U^{-1}$, where the $\dagger$ symbol represents the conjugate transpose of a matrix;
- $\det(U) = 1$;
- The eigenvalues of U are λ_1 and λ_2, with $|\lambda_i|^2 = 1$ for $i \in \{1, 2\}$;

When a unitary is applied to a qubit $q = |\psi\rangle$, it applies a *rotation* on the Bloch sphere around a certain axis. Such axis is always identified by the eigenvectors of unitary U, also called *eigenstates*. By definition, in fact, an eigenstate $|\psi_U\rangle$ of the unitary U is a state such that:

$$U |\psi_U\rangle = \lambda_U |U\rangle$$

where $\lambda_U \in \mathbb{C}$ is an *eigenvalue* of U related to the eigenstate $|\psi_U\rangle$.

In the gate-model, a unitary U is represented by a *gate*, which can be applied to a qubit q (represented by a *wire*). For this reason, we use gate and unitary in an interchangeable way.



Z-Pauli Gate The Pauli gate Z is called *phase-flip* operator and it is represented by the unitary matrix:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Using braket notation, the effects of the Z operator are the following ones:

$$Z |0\rangle = |0\rangle$$
$$Z |1\rangle = -|1\rangle$$

As you can see, $|0\rangle$ and $|1\rangle$ are by definition the eigenstates of Z operator, respectively with eigenvalue 1 and -1. In fact, $|0\rangle$ and $|1\rangle$ identifies the z-axis in Bloch sphere representation, therefore the Z-Pauli operator represents a rotation of π around the z-axis.

X-Pauli Gate The Pauli gate X is also called *quantum not*, since it swaps simply the state $|0\rangle$ into $|1\rangle$, and vice-versa. It is identified by the unitary matrix:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The eigenstates of X-Pauli operator are the states:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

and their respective eigenvalues are 1 and -1. In fact, we have that:

$$X |+\rangle = X \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) = \frac{X |0\rangle + X |1\rangle}{\sqrt{2}} = \frac{|1\rangle + |0\rangle}{\sqrt{2}} = |+\rangle$$
$$X |-\rangle = X \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) = \frac{X |0\rangle - X |1\rangle}{\sqrt{2}} = \frac{|1\rangle - |0\rangle}{\sqrt{2}} = -|-\rangle$$

$|+\rangle$ and $|-\rangle$ identifies the x-axis in Bloch sphere representation, therefore the X-Pauli operator represents a rotation of π around the x-axis.

Y-Pauli Gate The Pauli gate Y is identified by the unitary matrix:

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

In braket notation, the effects of Y-Pauli operator on the basis states are the following ones:

$$Y|0\rangle = i|1\rangle$$
$$Y|1\rangle = -i|0\rangle$$

The eigenstates of Y-Pauli operator are the states:

$$|+i\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \quad |-i\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$$

and their respective eigenvalues are 1 and -1. In fact, we have that:

$$Y|+i\rangle = Y\left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}\right) = \frac{Y|0\rangle + iY|1\rangle}{\sqrt{2}} = \frac{i|1\rangle + \cancel{(-i)^2}|0\rangle}{\sqrt{2}} = |+i\rangle$$
$$Y|-i\rangle = Y\left(\frac{|0\rangle - i|1\rangle}{\sqrt{2}}\right) = \frac{Y|0\rangle - iY|1\rangle}{\sqrt{2}} = \frac{i|1\rangle - \cancel{(-i)^2}|0\rangle}{\sqrt{2}} = -|-i\rangle$$

$|+i\rangle$ and $|-i\rangle$ identifies the y-axis in Bloch sphere representation, therefore the Y-Pauli operator represents a rotation of π around the y-axis.

Hadamard Gate Hadamard gate H plays the fundamental role of creating *superposition* when applied to the basis states $|0\rangle$ and $|1\rangle$. In fact we have that H is described by the unitary matrix:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and, in braket notation, we have that:

$$H|0\rangle = |+\rangle \qquad H|1\rangle = |-\rangle$$
$$H|+\rangle = |0\rangle \qquad H|-\rangle = |1\rangle$$

As for any single-qubit unitary, H corresponds to a rotation around a certain axis up to a global phase. To identify the rotation axis, one starts from the eigenvector of H, then normalize it to get the corresponding eigenstate, and convert it to the Bloch representation to obtain a vector that points along the rotation axis. The eigenvectors are: [1]

$$v = \left(\left(1 \pm \sqrt{2}\right)t, t\right), \quad \lambda = \pm 1$$

To turn this eigenvector into a quantum state, we must choose the free parameter t so that the state satisfies the amplitude normalization constraint, i.e., it has unit norm $\langle v_+|v_+\rangle = 1$. If the amplitudes are complex, the normalization factor is $\sqrt{|a|^2 + |b|^2}$, if they are real the normalization factor coincides with the Euclidean norm. After normalization, the eigenvector with $\lambda = +1$ corresponds to this eigenstate $|v_+\rangle$:

$$|v_+\rangle = \frac{\left(1+\sqrt{2}\right)|0\rangle + |1\rangle}{\sqrt{\left(1+\sqrt{2}\right)^2 + 1^2}} = \frac{1}{\sqrt{4+2\sqrt{2}}}\left(\left(1+\sqrt{2}\right)|0\rangle + |1\rangle\right)$$

The normalization applies a coefficient to the eigenvector, notice that this normalized $|v_+\rangle$ is still an eigenvector, we have simply chosen the specific t that makes this a quantum state. Let's check that $|v_+\rangle$ is an eigenstate of H.

[1] A nice symbolic calculator online https://matrixcalc.org/vectors.html

For brevity, we define a scalar γ:

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{4+2\sqrt{2}}} \\
|v_+\rangle &= \gamma\left(\left(1+\sqrt{2}\right)|0\rangle + |1\rangle\right) \\
H|v_+\rangle &= \gamma H\left(\left(1+\sqrt{2}\right)|0\rangle + |1\rangle\right) \\
&= \gamma\left(\left(1+\sqrt{2}\right)H|0\rangle + H|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\gamma\left(\left(1+\sqrt{2}\right)(|0\rangle + |1\rangle) + |0\rangle - |1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\gamma\left(\left(2+\sqrt{2}\right)|0\rangle + \sqrt{2}|1\rangle\right) \\
&= \gamma\left(\left(1+\sqrt{2}\right)|0\rangle + |1\rangle\right) \\
&= |v_+\rangle
\end{aligned}$$

We can apply the same process to the eigenvector associated to the negative eigenvalue:

For brevity, we define a scalar η which corresponds to the required normalization:

$$\begin{aligned}
\eta &= \frac{1}{\sqrt{4-2\sqrt{2}}} \\
|v_-\rangle &= \eta\left(\left(1-\sqrt{2}\right)|0\rangle + |1\rangle\right) \\
H|v_-\rangle &= \eta H\left(\left(1-\sqrt{2}\right)|0\rangle + |1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\eta\left(\left(1-\sqrt{2}\right)(|0\rangle + |1\rangle) + |0\rangle - |1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\eta\left(\left(2-\sqrt{2}\right)|0\rangle - \sqrt{2}|1\rangle\right) \\
&= \eta\left(\left(\sqrt{2}-1\right)|0\rangle - |1\rangle\right) \\
&= -\eta\left(\left(1-\sqrt{2}\right)|0\rangle + |1\rangle\right) \\
&= -|v_-\rangle
\end{aligned}$$

2 Phase Kickback

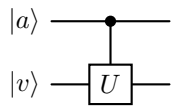
Phase kickback is a simple but useful phenomenon that is leveraged by various quantum algorithms. If we apply the unitary U to a state $|\psi\rangle$ which is one of its eigenstates, we obtain:

$$U|v\rangle = \lambda|v\rangle = e^{i\phi}|v\rangle$$

with $e^{i\phi}$ the eigenvalue corresponding to the eigenstate $|v\rangle$. The coefficient $\lambda = e^{i\phi}$ multiplied to the state is a *phase factor* and is, in general, a complex value with unit modulus. If U is real-valued and symmetric, its eigenvalues are real, hence equal to ± 1. Note that U refers to a generic unitary, and therefore includes all the quantum gates as well as combinations of them. Indeed, if you compute the product of all gates applied to one or more qubits you can represent an entire quantum circuit with a single unitary matrix.

In this example the phase is a coefficient multiplied to the whole system, which is called a *global* phase, and is impossible to observe. This tells us that there are cases in which applying a gate to a state does not make observable changes to the state! Indeed, states $-|-\rangle$ and $|-\rangle$ are indistinguishable.

In some quantum algorithms, such as the Deutsch algorithm, the creation of a phase when applying a quantum gate to one of its eigenvectors is used as part of the computation. To perform useful computations with this behaviour we first need to make the phase observable, which means we need to transform it from a global phase to a *relative* phase. Relative phases are applied only to some of the states in a superposition. Intuitively, if the goal is to apply the phase only to part of the states in a superposition we could use a controlled gate C_U to apply the unitary U or not based on a control qubit. Note that state $|v\rangle$ should be an eigenstate of the unitary U and not of the unitary C_U representing the controlled operation. Let's consider this circuit:

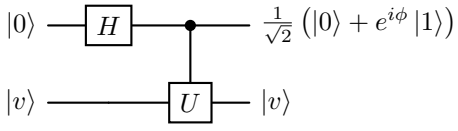


$$C_U = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}$$

What is the final state if $a = 0, 1$?

$$C_U|0, v\rangle = |0, v\rangle$$
$$C_U|1, v\rangle = |1\rangle \otimes U|v\rangle = |1\rangle \otimes e^{i\phi}|v\rangle = e^{i\phi}|1, v\rangle$$

No luck, we have either no phase or another global phase. The solution is to put the control qubit in a superposition, to create a state in which that phase is relative. Let's consider this circuit:



If we run the calculation we can see that the relative phase is applied not to the target qubit, $|v\rangle$, but rather, and counter-intuitively, to the *control* qubit. The phase kickback is due to the controlled gate transforming $|v\rangle$ into $e^{i\phi}|v\rangle$, the factor being the corresponding eigenvalue. This factor is then factored-out and ends up affecting the control qubit. See the following:

$$C_U(H \otimes I)|0, v\rangle = C_U \frac{1}{\sqrt{2}}(|0, v\rangle + |1, v\rangle) = \frac{1}{\sqrt{2}}\left(|0, v\rangle + |1\rangle \otimes e^{i\phi}|v\rangle\right)$$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right) \otimes |v\rangle$$

This phenomenon is called phase kickback because the phase $e^{i\phi}$ created by the application of unitary U to the state $|v\rangle$ ends up as a relative phase on the control qubit. Phase kickback extends to powers of the controlled unitary, indeed if U is a unitary, so is U^n, and $|v\rangle$ will still be one of its eigenstates. In this

case the kicked-back phase will be a power of the same eigenvalue, equivalently a multiple of the phase angle:

$$UU\left|v\right\rangle = e^{i\phi}U\left|v\right\rangle = e^{2i\phi}\left|v\right\rangle$$

In the general case of a phase kickback we apply a controlled unitary C_U on its eigenstate $|v\rangle$. If we measure the final state directly in the computational basis, the relative phase is not observable:

$$C_U\left(H\otimes I\right)\left|0,v\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + e^{i\phi}\left|1\right\rangle\right)\otimes\left|v\right\rangle$$

$$P(q_0=0) = \left|\left\langle 0\right|\frac{1}{\sqrt{2}}\left(\left|0\right\rangle + e^{i\phi}\left|1\right\rangle\right)\right\rangle\Big|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \left(\frac{1}{\sqrt{2}}\right)^\dagger\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

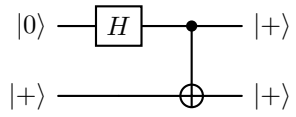
$$P(q_0=1) = \left|\left\langle 1\right|\frac{1}{\sqrt{2}}\left(\left|0\right\rangle + e^{i\phi}\left|1\right\rangle\right)\right\rangle\Big|^2 = \left|\frac{e^{i\phi}}{\sqrt{2}}\right|^2 = \left(\frac{e^{i\phi}}{\sqrt{2}}\right)^\dagger\left(\frac{e^{i\phi}}{\sqrt{2}}\right) = \frac{e^{-i\phi}}{\sqrt{2}}\frac{e^{i\phi}}{\sqrt{2}} = \frac{1}{2}$$

Notice how the probability of measuring 0/1 is 50%, so the relative phase has no impact on this measurement. To measure the relative phase we need to rotate it via additional gates but how to do this depends on what we aim to achieve.

2.1 Some useful examples of phase kickback

2.1.1 CNOT

If we apply a controlled not, C_X on $|{+}{+}\rangle$ we see that the state is unchanged. This is because the state $|+\rangle$ is an eigenvector of X with eigenvalue 1, hence the phase kicked back is $\lambda = e^{i0} = 1$.



$$C_X\left|+,+\right\rangle = C_X\frac{1}{\sqrt{2}}\left(\left|0,+\right\rangle + \left|1,+\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|0,+\right\rangle + \left|1,+\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right)\otimes\left|+\right\rangle = \left|+,+\right\rangle$$

If we apply C_X on $|{+}{-}\rangle$ or $|{-}{-}\rangle$ we see that a *negative* phase is kicked back. This is because the state $|-\rangle$ is an eigenvector of X with eigenvalue -1, hence the phase kicked back is $\lambda = e^{i\pi} = -1$.



Figure 1: C_X on $|{+}{-}\rangle$ and on $|{-}{-}\rangle$

$$C_X\left|+,-\right\rangle = C_X\frac{1}{\sqrt{2}}\left(\left|0,-\right\rangle + \left|1,-\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|0,-\right\rangle - \left|1,-\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle - \left|1\right\rangle\right)\otimes\left|-\right\rangle = \left|-,-\right\rangle$$

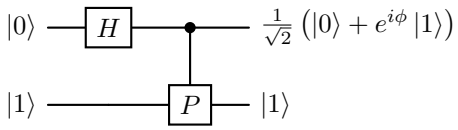
$$C_X\left|-,-\right\rangle = C_X\frac{1}{\sqrt{2}}\left(\left|0,-\right\rangle - \left|1,-\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|0,-\right\rangle + \left|1,-\right\rangle\right) = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right)\otimes\left|-\right\rangle = \left|+,-\right\rangle$$

Note that in all previous cases the application of a Hadamard gate on the control qubit will result in it assuming state $|0\rangle$ or $|1\rangle$ with 100% probability and therefore will be readable with one single measurement.

2.1.2 Controlled Phase

Applying a controlled phase gate has a similar effect. The $P(\phi)$ unitary has two eigenvectors: $|0\rangle$ with eigenvalue 1 and $|1\rangle$ with eigenvalue $e^{i\phi}$. Note that $P(\pi) = Z$, $P(\frac{\pi}{2}) = S$, and $P(\frac{\pi}{4}) = T$.

$$P(\phi) = \begin{pmatrix}1 & 0\\ 0 & e^{i\phi}\end{pmatrix} \quad C_P = \begin{pmatrix}I & 0\\ 0 & P\end{pmatrix}$$



$$C_P|+,0\rangle = C_P\frac{1}{\sqrt{2}}(|0,0\rangle + |1,0\rangle) = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\otimes|0\rangle = |+,0\rangle$$
$$C_P|+,1\rangle = C_P\frac{1}{\sqrt{2}}(|0,1\rangle + |1,1\rangle) = \frac{1}{\sqrt{2}}\left(|0,1\rangle + e^{i\phi}|1,1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\phi}|1\rangle\right)\otimes|1\rangle$$

2.1.3 Controlled Hadamard

In case of a controlled Hadamard gate, as we have seen in the previous lesson, it is less obvious how to construct a state that is an eigenvector of H.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \quad C_H = \begin{pmatrix}I & 0\\ 0 & H\end{pmatrix}$$

For the purpose of this example, we will choose the eigenstate of H associated to the negative eigenvalue, in order for the phase kickback to be visible. From the past lecture we know that the eigenstate is:

$$|v_-\rangle = \frac{1}{\sqrt{4-2\sqrt{2}}}\left(\left(1-\sqrt{2}\right)|0\rangle + |1\rangle\right)$$

We have already proved that the application of H to this eigenstate causes the creation of a negative phase. We use this result and apply the controlled Hadamard operation on a $|+,v_-\rangle$ state to see the phase kickback in action

$$\begin{aligned}C_H|+,v_-\rangle &= C_H\frac{1}{\sqrt{2}}(|0,v_-\rangle + |1,v_-\rangle) = \frac{1}{\sqrt{2}}(|0,v_-\rangle + |1\rangle\otimes H|v_-\rangle)\\ &= \frac{1}{\sqrt{2}}(|0,v_-\rangle - |1,v_-\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\otimes|v_-\rangle = |-,v_-\rangle\end{aligned}$$